JOURNAL OF COMBINATORIAL THEORY, Series A 33, 273–286 (1982)

Bijective Proofs of Some Classical Partition Identities

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Communicated by the Managing Editors

Received September 7, 1981; revised November 11, 1981.

A bijective proof of a general partition theorem is given which has as direct corollaries many classical partition theorems due to Euler, Glaisher, Schur, Andrews, Subbarao, and others. It is shown that the bijective proof specializes to give bijective proofs of these classical results and moreover the bijections which result often coincide with bijections which have occurred in the literature. Also given are some sufficient conditions for when two classes of words omitting certain sequences of words are in bijection.

INTRODUCTION

In this paper, we shall give a bijective proof of a simple yet amazingly general partition theorem. This general partition theorem has as direct corollaries many classical partition theorems including results due to Euler, Glaisher, Schur, Andrews, Subbarao, and others. Our bijective proof of our general partition theorem immediately gives bijective proofs of all such classical theorems and remarkably we shall show that in several cases the bijection which results coincides with the classical bijections found in the literature.

Our work here is based on combining two ideas that have occurred in the literature. Andrews' theory of partition ideals of order 1 in [1] gave a general partition theorem which had as direct corollaries many classical partition theorems. Later, Cohen [5] developed a theory of P.I.E. sums and was able to give a simple general partition theorem which extended Andrews' results. Our general partition theorem is but a slight extension of Cohen's result. Neither Andrews nor Cohen, however, gave bijective proofs of their results. To give a bijective proof of our result, we use a new and important technique, due to Garsia and Milne [7], for building bijections out of certain

* Partially supported by National Science Foundation Grant MCS79-3406.

0097-3165/82/060273–14$02.00/0

pairs of involutions. In Section 1 we shall briefly outline Garsia and Milne's technique and then in Section 2 we shall prove our general theorem and examine the various special cases. In Section 3, we shall prove a result similar to our general partition theorem which gives sufficient conditions for when two classes of words omitting certain sequences of words are in bijection.

1. The Involution Principle

In this section we shall briefly outline a fundamental method for constructing bijections out of certain pairs of involutions discovered by Garsia and Milne [7] who used the method to give the first bijective proof of the famous Rogers–Ramanujan identities. Assume we have two disjoint finite spaces A and B and both A and B are further partitioned into *positive* and *negative* parts, $A = A^+ \cup A^-$ and $B = B^+ \cup B^-$. Assume we have a sign preserving bijection f from A to B, i.e., $f(A^+) = B^+$ and $f(A^-) = B^-$. Next assume we have a pair of what we term ***sign reversing bijections*** α from A onto A and β from B onto B, with positive fixed points. That is, we assume $\alpha: A \to A$ is a bijection and for all $a \in A$ either (i) $\alpha(a) = a$ and $a \in A^+$ or (ii) $\alpha(a) \neq a$ in which case $a \in A^+$ implies $\alpha(a) \in A^-$ and $a \in A^-$ implies $\alpha(a) \in A^+$ and similarly for β. Let F_α and F_β denote the fixed point sets of α and β, respectively. Note that we immediately have that $|F_\alpha| = |F_\beta|$ since by α we have $|A^+| - |F_\alpha| = |A^-|$, by f we have $|A^-| = |B^-|$ and $|A^+| = |B^+|$, and by β we have $|B^+| - |F_\beta| = |B^-|$. The fundamental observation of Garsia and Milne is that a direct bijection between F_α and F_β can be constructed out of α, β, and f. Let $\alpha^* = f \circ \alpha$ and $\beta^* = f^{-1} \circ \beta$ so that α^* maps A one–one onto B and β^* maps B one-one onto A. Now for any fixed point, $a \in F_\alpha$, we form a sequence $a = a_1, b_1, a_2, b_2, \ldots$, by first applying α^*, then β^*, then α^*, etc., i.e., $a_{i+1} = \beta^*(b_i)$ and $b_i = \alpha^*(a_i)$ for $i = 0, 1, \ldots$. We call $a_1, b_1, \ldots$ the *iterated* (α, β)*-sequence* associated with a. There are two basic facts to establish about such sequences:

(I) If $a \in F_\alpha$, then there is a least n, denoted by n_a, such that $b_n \in F_\beta$.

(I) is easily established by showing by induction that if there is no such n, then $a_1, b_1, a_2, b_2, \ldots$, are all pairwise distinct violating the finiteness of A and B. Having established (I), we can show the following by appealing to the fact that α^* and β^* are one–one:

(II) if $a, a' \in F_\alpha$ with iterated (α, β)-sequences $a = a_1, b_1, \ldots, a_{n_a}, b_{n_a}$ and $a' = a'_1, b'_1, \ldots, a'_{n_{a'}}, b'_{n_{a'}}$, respectively, then $a \neq a'$ implies $b_{n_a} \neq b'_{n_{a'}}$.

Now given (I), we can define a map $I\langle \alpha, \beta, f\rangle: F_\alpha \to F_\beta$, which we shall call the *iterated* (α, β)*-map*, by $I\langle \alpha, \beta, f\rangle(a) = b_{n_a}$ for all $a \in F_\alpha$. By (II), it follows that $I\langle\alpha,\beta,f\rangle$ is one–one and by a symmetrical argument, it is easy to show that $I\langle\alpha,\beta,f\rangle$ is onto. Thus we have

THEOREM 1 (Garsia–Milne [7]). *Let* $A=A^+\cup A^-$, $B=B^+\cup B^-$, $f\colon A\to B$, $\alpha\colon A\to A$, *and* $\beta\colon B\to B$ *be described as above, then the iterated* (α,β)*-map* $I\langle\alpha,\beta,f\rangle$ *is a bijection between* F_α*and* F_β.

Remark. There are many applications of Theorem 1. See e.g., [7, 10–12]. In all of these applications, α and β are involutions. In fact, in Garsia and Milne's original paper [7], they state Theorem 3.1 in a slightly different setting and only for involutions although their proof remains unchanged for sign reversing bijections. Thus, we shall sometimes refer to an application of Theorem 1 as an application of the involution principle.

2. The General Partition Theorem

We think of a partition Π of n, written $\Pi\vdash n$, as a multiset $\Pi=\{n_1*a_1,\ldots,n_k*a_k\}$, where n_1*a_i means that a_i occurs n_i times in Π and let $|\Pi|=\sum_{i=1}^k n_ia_i$. Many classical partition identities concern sets of partitions Π which fail to contain any elements in a certain sequence of multisets $\mathcal{A}=\langle A_i\rangle_{i\in\omega}$. For example, the set of partitions with distinct parts is the set of partitions which do not contain any multisets in the sequence $\langle\{1,1\},\{2,2\},\ldots\rangle$. If $\mathcal{A}=\langle A_i\rangle_{i\in\omega}$ is a sequence of multisets and Π is a partition, we define $S_{\mathcal{A}}(\Pi)$ to be the set of indices i such that $A_i\subseteq\Pi$, i.e., $S_{\mathcal{A}}(\Pi)=\{i\mid A_i\subseteq\Pi\}$. $P_n(\mathcal{A})$ will denote the set of partitions of n which do not contain any of the multisets A_i. Thus $P_n(\mathcal{A})=\{\Pi\mid\Pi\vdash n$ and $S_{\mathcal{A}}(\Pi)=\phi\}$. Given two multisets A_1 and A_2, $A_1\cup A_2$ is the multiset such that the number of times an element **a** occurs in A is the maximum of the number of times **a** occurs in A_1 and the number of times **a** occurs in A_2. For example, $\{1,1,1,2,2\}\cup\{1,2,2,2,3\}=\{1,1,1,2,2,2,3\}$ and $S_{\mathcal{A}}(\{1,1,1,2,2,2,3\})=\{1,2\}$, where $\mathcal{A}=\langle\{1,1\},\{2,2\},\ldots\rangle$.

Andrews developed the theory of what he terms partition ideals of order 1 and proved a general theorem [1, Theorem 8.4] concerning such partition ideals which yields many classical partition theorems as special cases. Cohen proved a simple and elegant theorem [5, Theorem 7] which essentially contained Andrews' theorem as a special case and hence yielded all the corollaries of Andrews' theorem as a special case and many more. We shall show that Cohen's and hence Andrews' theorems can be given a general bijective proof using the methods given by Garsia and Milne in [7] and that the bijections that result in many of the special cases are the classical bijections. Actually, the next result is a slight generalization of Cohen's theorem but is more or less implicit in Cohen's generally theory of P.I.E. sums.

THEOREM 2. *Suppose* $\mathcal{A} = \{A_i\}_{i\in\omega}$ *and* $\mathcal{B} = \{B_i\}_{i\in\omega}$ *are sequences of distinct nonempty multisets such that for all finite sets* $S \subseteq \omega$, $|\bigcup_{i\in S} A_i| = |\bigcup_{i\in S} B_i|$, *then* $|P_n(\mathcal{A})| = |P_n(\mathcal{B})|$.

Remark. The easiest way to ensure that $|\bigcup_{i\in S} A_i| = |\bigcup_{i\in S} B_i|$ for all finite sets S is to have (a) $\mathcal{A}$ and $\mathcal{B}$ be sequences of pairwise disjoint multisets and (b) $|A_i| = |B_i|$ for all i. Cohen's Theorem 7 [5] which he calls the disjoint case of P.I.E. sums has as hypotheses (a) and (b) above plus the assumption that the $|A_i|$'s are all distinct which is unnecessary. We shall refer to hypotheses (a) and (b) as the disjoint case. Andrews' Theorem 8.4 [1], while not expressed in this language, is the disjoint case where each of the multisets of $\mathcal{A}$ and $\mathcal{B}$ are of the form $\{n * k\}$ for some k.

Proof. We shall use the involution principle (Theorem 1) to construct a bijection between $P_n(\mathcal{A})$ and $P_n(\mathcal{B})$ for any fixed n. For the space A, we consider $\{(\Pi, S) \mid \Pi \vdash n \text{ and } S \subseteq S_{\mathcal{A}}(\Pi)\}$, where the sign of a pair $(\Pi, S) \in A$ is $(-1)^{|S|}$. Similarly the space $B = \{(\Pi, S) \mid \Pi \vdash n \text{ and } S \subseteq S_{\mathcal{B}}(\Pi)\}$. The involutions α and β are quite simple. For a given partition Π, let $a_\Pi = \max(S_{\mathcal{A}}(\Pi))$ if $S_{\mathcal{A}}(\Pi) \neq \phi$ and $b_\Pi = \max(S_{\mathcal{B}}(\Pi))$ if $S\mathcal{B}(\Pi) \neq \phi$. Then if $S_{\mathcal{A}}(\Pi) \neq \phi$, we define

$$\begin{aligned} \alpha(\Pi, S) &= (\Pi, S - \{a_\Pi\}), && \text{if } a_\Pi \in S, \\ &= (\Pi, S \cup \{a_\Pi\}), && \text{if } a_\Pi \notin S, \end{aligned}$$

and if $S_{\mathcal{A}}(\prod) = \phi$, we define $\alpha(\Pi, \phi) = (\Pi, \phi)$. Similarly, if $S_{\mathcal{B}}(\Pi) \neq \phi$, we define

$$\begin{aligned} \beta(\Pi, S) &= (\Pi, S - \{b_\Pi\}), && \text{if } b_\Pi \in S, \\ &= (\Pi, S \cup \{b_\Pi\}), && \text{if } b_\Pi \notin S, \end{aligned}$$

and if $S_{\mathcal{B}}(\Pi) = \phi$, we define $\beta(\Pi, \phi) = (\Pi, \phi)$. Clearly, α is a sign-reversing involution on A with fixed point set $F_\alpha = \{(\Pi, \phi) \mid \Pi \in P_n(\mathcal{A})\}$ and β is a sign-reversing involution on B with fixed point set $F_\beta = \{(\Pi, \phi) \mid \Pi \in P_n(\mathcal{B})\}$. Finally, we define a sign preserving map $f\colon A \to B$ by defining for $(\Pi, S) \in A$, $f(\Pi, S) = (\lambda, S)$, where $\lambda = [\Pi - (\bigcup_{i\in S} A_i)] \cup [\bigcup_{i\in S} B_i]$. Thus by Theorem 1, the iterated (α, β)-map gives a bijection between $P_n(\mathcal{A}) \times \{\phi\}$ and $P_n(B) \times \{\phi\}$ which may be regarded as a bijection between $P_n(\mathcal{A})$ and $P_n(\mathcal{B})$. ∎

We should also remark that Andrews' Theorem 8.4 [1] stated a converse of Theorem 2 in his case while Cohen stated no converse to his Theorem 7. The obvious converse to Theorem 2 fails; see the discussion following Corollary 2.3 for a counterexample. In the disjoint case, however, there is a converse to Theorem 2.

THEOREM 3. *Suppose $\mathcal{A}=\{A_i\}_{i\in\omega}$ and $\mathcal{B}=\{B_i\}_{i\in\omega}$ are sequences of distinct nonempty multisets. Then $|P_n(\mathcal{A})|=|P_n(\mathcal{B})|$ for all n iff the sequences $\{|A_i|\}_{i\in\omega}$ and $\{|B_i|\}_{i\in\omega}$ are rearrangements of each other.*

Proof. The *if* part of Theorem 3 follows from Theorem 2. For the *only if* part, we define for each partition or multiset $\Pi=\{n_1 * a_1,\dots, n_k * a_k\}$, a monomial $x(\Pi)=x_{a_1}^{n_1}\cdots x_{a_k}^{n_k}$, where $x_1, x_2,\dots,$ are indeterminates. Then by a usual inclusion–exclusion argument, it is easy to see that since the sequences of multisets $\{A_i\}$ and $\{B_i\}$ are pairwise disjoint that:

$$\sum_{n\geqslant 0}\sum_{\Pi\in P_n(\mathcal{A})} x(\Pi)=\prod_{n=1}^{\infty}\left(\frac{1}{1-x_n}\right)\prod_{k=0}^{\infty}(1-x(A_k)) \tag{2.1}$$

and

$$\sum_{n\geqslant 0}\sum_{\Pi\in P_n(\mathcal{B})} x(\Pi)=\prod_{n=1}^{\infty}\frac{1}{1-x_n}\prod_{k=0}^{\infty}(1-x(B_k)). \tag{2.2}$$

Now replacing x_i by q^i in (2.1) and (2.2) we have

$$\sum_{n\geqslant 0}|P_n(\mathcal{A})|\,q^n=\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)\prod_{k=0}^{\infty}(1-q^{|A_k|}) \tag{2.3}$$

and

$$\sum_{n\geqslant 0}|P_n(\mathcal{B})|\,q^n=\prod_{n=1}^{\infty}\left(\frac{1}{1-q^n}\right)\prod_{k=0}^{\infty}(1-q^{|B_k|}). \tag{2.4}$$

Now if $|P_n(\mathcal{A})|=|P_n(\mathcal{B})|$ for all n, then multiplying (2.3) and (2.4) by $\prod_{n=1}^{\infty}(1-q_n)$, we have

$$\prod_{k=0}^{\infty}(1-q^{|A_k|})=\prod_{k=0}^{\infty}(1-q^{|B_k|}) \tag{2.5}$$

which implies the sequences $\{|A_i|\}_{i\in\omega}$ and $\{|B|_i\}_{i\in\omega}$ are rearrangements of each other. ∎

Next our goal is to list some of the classical partition theorems which follow from Theorem 2 and to analyze the bijection given by our proof and show that in many cases the bijection of Theorem 2 is identical with the classical bijection. Our first four corollaries were listed by both Andrews and Cohen.

COROLLARY 2.1 (Euler [6]). *The number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

COROLLARY 2.2 (Glaisher [8]). *The number of partitions of n with no* *part divisible by d equals the number of partitions of n in which no part is repeated d or more times.*

Of course, Euler's result is the special case of Glaisher's result, where $d=2$. To prove Corollary 2.2 from Theorem 2, note that the number of partitions of n with no parts divisible by d equals $P_n(\mathcal{A}_{G,d})$, where $\mathcal{A}_{G,d} = \langle \{d\}, \{2d\}, \{3d\},... \rangle$ and the number of partitions of n with no part repeated d or more times equals $P_n(\mathcal{B}_{G,d})$, where

$$\mathcal{B}_{G,d} = \langle \underbrace{\{1,..., 1\}}_{d\,\text{times}} \underbrace{\{2,..., 2\}}_{d\,\text{times}},... \rangle = \langle \{d*1\}, \{d*2\}, \{d*3\},... \rangle.$$

Glaisher [8] constructed the following bijection: $\theta: P_n(\mathcal{A}_{G,d}) \to P_n(\mathcal{B}_{G,d})$. Given $\Pi \in P_n(\mathcal{A}_{G,d})$, we write $\Pi = \{n_1 * a_1, n_2 * a_2,..., n_k * a_k\}$, where $d \nmid a_i$ for any i. Now for each i we write the d-ary expansion of $n_i = e_0^i d^0 + e_1^i d^1 + \cdots + e_{l_i}^i d^{l_i}$, where $o \leqslant e_j^i \leqslant d-1$ for $j = 1,..., l_i$. Then $\theta(\Pi)$ is the partition that results by replacing each sequence of parts $n_i * a_i$ in Π by the sequence of parts $\{e_0^i * d^0 a_i, e_1^i * d^1 a_i,..., e_{l_i} * d^{l_i} a_i\}$. It is easy to check that θ is a bijection between $P_n(\mathcal{A}_{G,d})$ and $P_n(\mathcal{B}_{G,d})$.

What is remarkable is that θ is exactly the bijection given by our general bijective proof of Theorem 2.

THEOREM 4. *The iterated (α, β)-map given in Theorem 2 between $P_n(\mathcal{A}_{G,d})$ and $P_n(\mathcal{B}_{G,d})$ is the Glaisher bijection θ for any fixed d.*

Proof. We know by the proof of Theorem 2 that if we start with a pair $(\Pi, \phi) \in F_\alpha$, where $\Pi = \{n_1 * a_1,..., n_k * a_k\}$, where $d \nmid a_i$ for any i and apply the iterated (α, β)-map that we must end with a partition $(\lambda, \phi) \in F_\beta$, where λ has no part repeated more than d times. Let $(\Pi, \phi) = (\Pi_0, S_0), (\Pi_1, S_1),..., (\Pi_k, S_k) = (\lambda, \phi)$ be the sequence of pairs that results in applying the iterated (α, β)-map to (Π, ϕ), i.e., $(\Pi_1, S_1) = \alpha(\Pi, \phi)$, $(\Pi_2, S_2) = f \circ \alpha(\Pi, \phi)$, $(\Pi_3, S_3) = \beta \circ f \circ \alpha(\Pi, \phi)$, $(\Pi_4, S_4) = f^{-1} \circ \beta \circ f \circ \alpha(\Pi, \phi)$, etc. Then for any given i, the only change between Π_i and Π_{i+1} occurs when either $(\Pi_{i+1}, S_{i+1}) = f(\Pi_i, S_i)$ or $(\Pi_{i+1}, S_{i+1}) = f^{-1}(\Pi_i, S_i)$. In the first case all that happens is that certain parts of Π_i of the form dk are replaced by d parts of size k and in the latter case d parts of Π_i of a fixed size k are replaced by one part of size dk for various k. Now since $d \nmid a_j$, it easily follows that for any j, the n_j parts of size a_j could only be coalesced into parts of size $d^i a_j$ for some i which appear in the final result λ. Moreover since no part occurs d or more times in λ, it follows that for each j the n_j parts of size a_j eventually are transformed into a sequence of parts in λ, $\varepsilon_0 * a_j$, $\varepsilon_1 * da_j$, $\varepsilon_2 * d^2 a_j,..., \varepsilon_r * d^r a_j$ for some r such that $0 \leqslant \varepsilon_i < d$ for $i = 0,..., r$ and $n_j a_j = \sum_{i=0}^r \varepsilon_i d^i a_j$. But then $n_j = \sum_{i=0}^r \varepsilon_i d^i$ so that $\sum_{i=0}^r \varepsilon_i d^i$ must be the unique d-ary expansion of n_j and hence $\lambda = \theta(\Pi)$. ∎

Remark. We should note that while the iterated (α, β)-map does give the Glaisher bijection between $P_n(\mathcal{A}_{G,d})$ and $P_n(\mathcal{B}_{G,d})$, the actual algorithm is torturously inefficient. For example, even in the simple case, where $d=2$, $n=18$, and where $\theta(\{1,1,3,3,5,5\}) = \{2,6,10\}$, the iterated (α,β)–map takes 14 iterations due to a tower of Hanoi-type effect in the algorithm. In Table I, we explicitly illustrate all the steps of the iterated (α,β)-map in his case. We note that the label at the top of each column gives the space in which the pair resides and the label between columns gives the map which sends one column to the next.

EXAMPLE 1. $d=2$, $n=18$, $\theta(\{1,1,3,3,5,5\}) = \{2,6,10\}$, $\mathcal{A} = \mathcal{A}_{G,2} = \langle\{2\},\{4\},\{6\},\{8\},\{10\},\ldots\rangle$, $A = \{(\Pi, S) \mid \Pi \vdash 18$ and $S \subseteq {}_{\mathcal{A}}(\Pi)$, $\mathcal{B} = \mathcal{B}_{G,2} = \langle\{1,1\},\{2,2\},\{3,3\},\{4,4\},\{5,5\},\ldots\rangle$, and $B = \{(\Pi, S) \mid \Pi \vdash 18$ and $S \subseteq S_{\mathcal{B}}(\Pi)\}$. For clarity, an ordered pair (Π, S) will be written Π/S. Note that we start with a fixed point of α and end with a fixed point of β.

COROLLARY 2.3 (Shur [13]).

(i) *The number of partitions of n into parts* $\equiv \pm 1$ mod 6 *equals*

(ii) *the number of partitions of n into distinct parts* $\equiv \pm 1$ mod 3.

Proof. Clearly, the partitions of type (i) are just $P_n(\mathcal{A}_s)$, where $\mathcal{A}_s = \langle\{2\},\{3\},\{4\},\{6\},\{8\},\{9\},\{10\},\{12\},\ldots\rangle$. For the partitions of type (ii), we must eliminate the multisets $\{1,1\}$, $\{2,2\}$, $\{3,3\}$,..., to leave only distinct parts and the multisets $\{3\}$, $\{6\}$, $\{9\}$,..., to leave only parts

TABLE I

	A	$\xrightarrow{\alpha}$	A	$\xrightarrow{f}$	B	$\xrightarrow{\beta}$	B	$\xrightarrow{f^{-1}}$
1	$1,1,3,3,5,5/\phi$	$=$	$1,1,3,3,5,5/\phi$		$1,1,3,3,5,5/\phi$		$1,1,3,3,5,5/\{5\}$	
2	$1,1,3,3,10/\{5\}$		$1,1,3,3,10/\phi$		$1,1,3,3,10/\phi$		$1,1,3,3,10/\{3\}$	
3	$1,1,6,10/\{3\}$		$1,1,6,10/\{3,5\}$		$1,1,3,3,5,5/\{3,5\}$		$1,1,3,3,5,5/\{3\}$	
4	$1,1,6,5,5/\{3\}$		$1,1,6,5,5/\phi$		$1,1,6,5,5/\phi$		$1,1,6,5,5/\{5\}$	
5	$1,1,6,10/\{5\}$		$1,1,6,10/\phi$		$1,1,6,10/\phi$		$1,1,6,10/\{1\}$	
6	$2,6,10/\{1\}$		$2,6,10/\{1,5\}$		$1,1,6,5,5/\{1,5\}$		$1,1,6,5,5/\{1\}$	
7	$2,6,5,5/\{1\}$		$2,6,5,5/\{1,3\}$		$1,1,3,3,5,5/\{1,3\}$		$1,1,3,3,5,5/\{1,3,5\}$	
8	$2,6,10/\{1,3,5\}$		$2,6,10/\{1,3\}$		$1,1,3,3,10/\{1,3\}$		$1,1,3,3,10/\{1\}$	
9	$2,3,3,10/\{1\}$		$2,3,3,10/\{1,5\}$		$1,1,3,3,5,5/\{1,5\}$		$1,1,3,3,5,5/\{1\}$	
10	$2,3,3,5,5/\{1\}$		$2,3,3,5,5/\phi$		$2,3,3,5,5/\phi$		$2,3,3,5,5/\{5\}$	
11	$2,3,3,10/\{5\}$		$2,3,3,10/\phi$		$2,3,3,10/\phi$		$2,3,3,10/\{3\}$	
12	$2,6,10/\{3\}$		$2,6,10/\{3,5\}$		$2,3,3,5,5/\{3,5\}$		$2,3,3,5,5/\{3\}$	
13	$2,6,5,5/\{3\}$		$2,6,5,5/\phi$		$2,6,5,5/\phi$		$2,6,5,5/\{5\}$	
14	$2,6,10/\{5\}$		$2,6,10/\phi$		$2,6,10/\phi$	$=$	$2,6,10/\phi$	

$\equiv \pm 1 \bmod 3$. Thus the partitions of type (ii) are just $P_n(\mathscr{B}_s)$, where $\mathscr{B}_S = \langle\{1, 1\}, \{3\}, \{2, 2\}, \{6\}, \{4, 4\}, \{9\}, \{5, 5\}, \{12\},...\rangle$. ∎

We should note that Corollary 2.3 represents the simplest part of Schur's results in [13]. Schur also shows that the partitions of type (i) or (ii) in Corollary 2.3 are equinumerous with:

(iii) The number of partitions of n in which the difference of any two parts is at least 3 and in which no consecutive multiples of 3 occur.

Now the fact that the number of partitions of type (i) equals the number of partitions of type (iii) does not follow from Theorem 2 and provides an examples that the obvious converse of Theorem 2 fails. That is, to leave only parts that differ by at least 3, we must eliminate the multisets of the form $\{i, i\}$, $\{i, i+1\}$, and $\{i, i+2\}$ for each $i \geqslant 1$ and to ensure there are no consecutive multiples of 3 we must eliminate the multisets of the form $\{3i, 3i+3\}$ for each $i \geqslant 1$. Thus, e.g., to be a partition of type (iii), Π cannot contain a certain multiset of size 5, namely $\{2, 3\}$, which has no counterpart in either $\mathscr{A}_S$ or $\mathscr{B}_S$. (We note that Bressoud [3] has given a bijection between the partitions of types (i) and (iii).)

Once again there is a Glaisher-type bijection between $P_n(\mathscr{A}_S)$ and $P_n(\mathscr{B}_S)$. Namely, given a partition in $P_n(\mathscr{A}_S)$, $\Pi = \{n_1 * a_1,..., n_k * a_k\}$, where $a_i \equiv \pm 1 \bmod 6$ for all i, we write each n_i in its binary expansion $n_i = 2^{\varepsilon_{0,i}} + 2^{\varepsilon_{1,i}} + \cdots + 2^{\varepsilon_{r,i}}$, where $0 \leqslant \varepsilon_{0,i} < \cdots < \varepsilon_{r,i}$, and replace the n_i parts of size a_i in Π by the sequence of parts $2^{\varepsilon_{0,i}}a_i, 2^{\varepsilon_{1,i}}a_i,..., 2^{\varepsilon_{r,i}}a_i$ to get a new partition $\Omega(\Pi)$. It is easy to check that $\Omega(\Pi)$ is in $P_n(\mathscr{B}_S)$ since if $a_i, a_j \equiv \pm 1 \bmod 6$, then $2^r a_i \equiv \pm 1 \bmod 3$ and $2^p a_i = 2^r a_j$ for some p and r implies $a_i = a_j$. Once again the iterated (α, β) – map of Theorem 2 gives the Ω bijection. That is, start with a pair $(\Pi, \phi) \in F_\alpha$ where $\Pi = \{n_1 * a_1,..., n_k * a_k\}$, where all $a_i \equiv \pm 1 \bmod 6$ and let $(\Pi, \phi) = (\Pi_0, S_0)$, $(\Pi_1, S_1),..., (\Pi_k, S_k) = (\lambda, \phi)$ be the sequence of pairs that results in applying the iterated (α, β)-map as defined in Theorem 4, where $(\lambda, \phi) \in P_n(\mathscr{B}_S)$. Again the only change from Π_i to Π_{i+1} occurs when either $(\Pi_{i+1}, S_{i+1}) = f(\Pi_i, S_i)$ or $(\Pi_{i+1} S_{i+1}) = f^{-1}(\Pi_i, S_i)$. In the first case, we replace certain parts of Π_i of the form $2i$ by two parts of size i and in the second case we replace certain pairs of parts of size i in Π_i by one part of size $2i$. It follows from our remark that the numbers of the form $2^r a_i$ and $2^p a_j$ are distinct for distinct a_i and $a_j \equiv \pm 1 \bmod 6$ that the n_i parts of size a_i in Π must eventually become a sequence of parts of $2^{\varepsilon_{0,i}}a_i,..., 2^{\varepsilon_{r,i}}a_i$ in λ. Since the parts of λ must be all distinct we must have that $n_i a_i = 2^{\varepsilon_{0,i}}a_i + \cdots + 2^{\varepsilon_{r,i}}a_i$, where $0 \leqslant \varepsilon_{0,i} < \cdots < \varepsilon_{r,i}$. Hence $2^{\varepsilon_{0,i}} + \cdots + 2^{\varepsilon_{r,i}}$ is the unique binary expansion of n_i and $\lambda = \Omega(\Pi)$.

Actually the easy half of Schur's theorem and Euler's theorem are special cases of the following theorem due to Andrews [2] which also is a corollary of Theorem 2:

COROLLARY 2.4 (Andrews [2]). *Let M_1 and M_2 be two sets of positive integers. Let $2M_1 = \{j \mid (j/2) \in M_1\}$. Then the number of partitions of n into dstinct parts taken from M_1 equals the number of partitions of n into parts taken from M_2 if $2M_1 \subseteq M_1$ and $M_2 = M_1 - 2M_1$.*

Proof. The partitions of n into parts taken from M_2 equals $P_n(\mathcal{A}_A)$, where $\mathcal{A}_A$ contains all singletons $\{i\}$, where $i \notin M_2$. The partitions of n into distinct parts taken from M_1 equals $P_n(\mathcal{B}_A)$, where $\mathcal{B}_A$ contains all singletons $\{i\}$, where $i \notin M_1$ and all doubletons $\{j, j\}$, where $j \in M_1$. For the correspondence between $\mathcal{A}_A$ and $\mathcal{B}_A$, note that the $\{i\}$ in $\mathcal{A}_A$, where $i \notin M_1$ correspond to the singletons $\{i\}$ in $\mathcal{B}_A$, where $i \notin M_1$ while the singletons $\{i\}$ in $\mathcal{A}_A$ such that $i \in M_1 - M_2$ correspond exactly to the doubletons $\{i/2, i/2\}$ in $\mathcal{B}_A$ since $M_1 - M_2 = 2M_1$. ∎

We note also that the Glaisher-type bijection Ω gives a bijective proof of Corollary 2.4. That is, if $\Pi = \{n_1 * a_1, ..., n_k * a_k\}$ is such that $a_i \notin M_2$ for all i, then the bijection Ω will replace each sequence of parts $n_i * a_i$ by $2^{\varepsilon_{0,i}} a_i, ..., 2^{\varepsilon_{r,i}} a_i$, where $2^{\varepsilon_{0,i}} + \cdots + 2^{\varepsilon_{r,i}} = n_i$ is the unique binary expansion of a_i. To see that Ω is in fact a bijection between $P_n(\mathcal{A}_A)$ and $P_n(\mathcal{B}_A)$ we observe two facts. First is the fact that since $M_2 \subseteq M_1$ and $2M_1 \subseteq M_1$, we have that $2^r a_i \in M_1$ for any r and any $a_i \in M_2$. Secondly, note that if $a_i, a_j \in M_2$, then $2^r a_i = 2^p a_j$ for some p, $r \geqslant 0$ implies $a_i = a_j$. For suppose $p < r$, then we have $2^{r-p} a_i = a_j \in M_2$ while $2^{r-p} a_i \in M_1$ which contradicts the fact that $M_2 = M_1 - 2M_1$. Similarly, we cannot have $r < p$, so $r = p$ and hence $a_i = a_j$. It thus follows that $\Omega(\Pi)$ will be in $P_n(\mathcal{B}_A)$, and clearly, Ω^{-1} exists so Ω is bijection between $P_n(\mathcal{A}_A)$ and $P_n(\mathcal{B}_A)$. Moreover, the two observations above are exactly what is required to show that the iterated (α, β)-map given by Theorem 2 between $P_n(\mathcal{A}_A)$ and $P_n(\mathcal{B}_A)$ is in fact just Ω by the same argument that followed Corollary 2.3. Thus we have the following:

THEOREM 5. *The iterated (α, β)-map given by Theorem 2 between $P_n(\mathcal{A}_A)$ and $P_n(\mathcal{B}_A)$ is the Glaisher-type bijection Ω, where $\mathcal{A}_A = \langle \{i\} \rangle_{i \notin M_2}$, $\mathcal{B}_A = \langle \{i\}, \{j, j\} \rangle_{i \notin M_1, j \in M_1}$ and $2M_1 \subseteq M_1$ and $M_2 = M_1 - 2M_1$.*

Subbarao generalized Andrews' result in [14] which also follows from Theorem 2.

COROLLARY 2.5 (Subbarao [14]). *Let S_1 and S_2 be two sets of positive integers. Let $dS_1 = \{j \mid (j/d) \in S_1\}$. Then the number of partitions of n into parts taken from S_2 equals the number of partitions of n with parts taken* *from* S_1, *where no part is repeated* d *or more times if* $dS_1 \subseteq S_1$ *and* $S_2 = S_1 - dS_1$.

Proof. The partitions of n with parts from S_2 equals $P_n(\mathcal{A}_{Su})$, where $\mathcal{A}_{Su}$ consists of all singletons $\{i\}$, where $i \notin S_2$. The partitions of n with parts from S_1, where no part is repeated d or more times equals $P_n(\mathcal{B}_{Su})$, where $\mathcal{B}_{Su}$ consists of all singletons $\{i\}$, where $i \notin S_1$ and all multisets $\{d * i\}$, where $i \in S_1$. Thus the singletons $\{i\}$ in $\mathcal{A}_{Su}$, where $i \notin S_1$ correspond to the singletons $\{i\}$ in $\mathcal{B}_{Su}$, where $i \notin S_1$ while the singletons $\{j\}$ in $\mathcal{A}_{Su}$, where $j \in S_1 - S_2$ correspond exactly to the multisets $\{d * (j/d)\}$, where $j \in S_1 - S_2$ since $S_1 - S_2 = dS_1$. ∎

As the reader must have guessed by now, the Glaisher bijection θ of Corollary 2.2 can be extended to give a bijection between $P_n(\mathcal{A}_{Su})$ and $P_n(\mathcal{B}_{Su})$, i.e., given $\Pi = \{n_1 * a_1, ..., n_k * a_k\}$ with $a_i \in S_2$ for all i, we let $\theta(\Pi)$ be the partition that results by replacing the parts $n_i * a_i$ by the sequence of parts $\varepsilon_{0,i} * a_i$, $\varepsilon_{1,i} * da_i, ..., \varepsilon_{r,i} * d^r a_i$, where $\varepsilon_{0,i} + \varepsilon_{1,i} d + \cdots + \varepsilon_{r,i} d^r = n_i$ is the unique d-ary expansion of n_i. By exactly the same type of argument that followed Corollary 2.4, we can show that θ is indeed a bijection between $P_n(\mathcal{A}_{Su})$ and $P_n(\mathcal{B}_{Su})$, and moreover, that θ is precisely the bijection given by the iterated (α, β)-map of Theorem 2.

Pairs (S_1, S_2) such that the number of partitions of n with parts taken from S_1 with no part repeated r or more times equals the number of partitions of n with parts taken from S_2 are called *Eulerian pairs of order r* by Subbarao. Of course the full statement of Andrews' results is that (M_1, M_2) is the Eulerian pair of order 2 iff $2M_1 \subseteq M_1$ and $M_2 = M_1 - 2M_1$ and the full statement of Subbarao's result is that (S_1, S_2) is the Eulerian pair of order r ($r \geqslant 2$), iff $rS_1 \subseteq S_1$ and $S_2 = S_1 - rS_1$. We note the *only if* parts of Andrews' and Subbarao's results are easy corollaries of Theorem 3. We also note that the Eulerian pair of order r, where $S_1 = \{1, r^n \mid n \in N\}$ and $S_2 = S_1 - rS_1 = \{1\}$ gives the uniqueness of the d-ary expansion which is yet another corollary of Theorem 2.

Of course one can use Theorem 2 to give literally an uncountable number of partitions theorems. We shall end this section by listing a few such theorems to illustrate the power of the hypotheses of Theorem 2.

For example, the fact that $10 = 1 + 9 = 2 + 8 = 3 + 7 = 4 + 6 = 5 + 5$ turns into the following partition theorem:

COROLLARY 2.6.

(i) *The number of partitions of n whose parts* $\equiv 1, 4 \bmod 5$ *do not differ by exactly* 8, *equals*

(ii) *the number of partitions of n whose parts* $\equiv 2, 3 \bmod 5$ *do not differ by exactly* 6, *equals*

(iii) *the number of partitions of n whose parts* $\equiv 2, 3 \bmod 5$ *do not differ by exactly* 4, *equals*

(iv) *the number of partitions of n whose parts* $\equiv 1, 4 \bmod 5$ *do not differ by exactly* 2, *equals*

(v) *the number of partitions of n with no repeated parts* $\equiv 0 \bmod 5$, *equals*

(iv) *the number of partitions of n with no multiples of* 10.

Proof. Let

$$\mathscr{A} = \langle \{1, 9\}, \{6, 14\}, \{11, 19\},\ldots \rangle,$$
$$\mathscr{B} = \langle \{2, 8\}, \{7, 13\}, \{12, 18\},\ldots \rangle,$$
$$\mathscr{C} = \langle \{3, 7\}, \{8, 12\}, \{13, 17\},\ldots \rangle,$$
$$\mathscr{D} = \langle \{4, 6\}, \{9, 11\}, \{14, 16\},\ldots \rangle,$$
$$\mathscr{E} = \langle \{5, 5\}, \{10, 10\}, \{15, 15\},\ldots \rangle,$$
$$\mathscr{F} = \langle \{10\}, \{20\}, \{30\},\ldots \rangle.$$

Then $P_n(\mathscr{A}) = P_n(\mathscr{B}) = P_n(\mathscr{C}) = P_n(\mathscr{E}) = P_n(\mathscr{F})$. ∎

The special case where $k = 3$ of the following two corollaries were given by Cohen in [5]:

COROLLARY 2.7. *For* $k \geqslant 3$ *the number of partitions of n with no odd multiples of* $\binom{k}{2}$ *equals the number of partitions of n with no consecutive parts* $\equiv 1, 2,\ldots, k-1 \bmod k$, *respectively.*

Proof. Let $\mathscr{A} = \langle \{1, 2,\ldots, k-1\}, \{k+1, k+2,\ldots, 2k-1\}, \{2k+1, 2k+2,\ldots, 3k-1\},\ldots \rangle$ and $\mathscr{B} = \langle \{k(k-1)/2\}, \{k(k-1) + k(k-1)/2\}, \{2k(k-1) + k(k-1)/2\},\ldots \rangle$, then $P_n(\mathscr{A}) = P_n(\mathscr{B})$. ∎

COROLLARY 2.8. *For* $k \geqslant 3$, *the number of partitions of n with no odd multiples of* $k^2 - k$ *equals the number of partitions of n with no consecutive repeated parts* $\equiv 1, 2,\ldots, k-1 \bmod k$, *respectively.*

Proof. Let $\mathscr{A} = \langle \{2 * 1, 2 * 2,\ldots, 2 * k-1\}, \{2 * k+1, 2 * k+2,\ldots, 2 * 2k-1\}, \{2 * 2k+1, 2 * 2k+2,\ldots, 2 * 3k-1\},\ldots \rangle$, $\mathscr{B} = \langle \{k^2 - k\}, \{2(k^2 - k) + k^2 - k\}, \{4(k^2 - k) + k^2 - k\},\ldots \rangle$. Then $P_n(\mathscr{A}) = P_n(\mathscr{B})$. ∎

The next few corollaries need the full hypothesis of Theorem 2 and do not follow from Cohen's disjoint case of P.I.E. sums.

COROLLARY 2.9. *The number of partitions of n with no consecutive even parts equals the number of partitions of n with no consecutive repeated parts.*

Proof. Let $\mathcal{A} = \langle \{2, 4\}, \{4, 6\}, \{6, 8\},...\rangle$ and $\mathcal{B} = \langle \{1, 1, 2, 2\}, \{2, 2, 3, 3\}, \{3, 3, 4, 4\},...\rangle$, then $P_n(\mathcal{A}) = P_n(\mathcal{B})$. ∎

COROLLARY 2.10. *The number of partitions of n with no consecutive parts repeated and no part repeated more than* 3 *times equals the number of partitions of n whose even parts differ by at least* 4.

Proof. Let $\mathcal{A}\langle \{1, 1, 1, 1\}, \{1, 1, 2, 2\}, \{2, 2, 2, 2\}, \{2, 2, 3, 3\}, \{3, 3, 3, 3\} \{3, 3, 4, 4\},...\rangle$ and $\mathcal{B} = \langle \{2, 2\}, \{2, 4\}, \{4, 4\}, \{4, 6\}, \{6, 6\}, \{6, 8\},...\rangle$, then $P_n(\mathcal{A}) = P_n(\mathcal{B})$. ∎

COROLLARY 2.11. *The number of partitions of n with no consecutive odd parts repeated and no odd part repeated more than* 3 *times equals the number of partitions of n whose parts* $\equiv 2 \bmod 4$ *differ by at least* 8.

Proof. Let $\mathcal{A} = \langle 1, 1, 1, 1\}\{1, 1, 3, 3\}, \{3, 3, 3, 3\}, \{5, 5, 5, 5\},...\rangle$ and $\mathcal{B} = \langle \{2, 2\}, \{2, 6\}, \{6, 6\}, \{6, 10\}, \{10, 10\},...\rangle$, *then* $P_n(\mathcal{A}) = P_n(\mathcal{B})$. ∎

3. BIJECTION BETWEEN CLASSES OF WORDS OMITTING CERTAIN WORDS

In this section we shall show that the same ideas used in the proof of Theorem 2 can be applied to give bijections between classes of words which fail to contain certain sequences of *forbidden* words. Such classes of words have been studied extensively in the literature (see Guibas and Odlyzko [9] for other references).

More formally, let $X = \{x_1,..., x_l\}$ be a finite alphabet and X^* denote the set of all finite words with letters from X. Given two words $u, v \in X^*$, we say u is a *factor* of v if v can be written as $v = w_1 u w_2$, where w_1 and w_2 are words (possibly empty) in X^*. We say that *u is a factor of v starting at i* if the length of w_1, $l(w_1)$, equals $i - 1$. Let $\mathcal{A} = \langle \alpha_1, \alpha_2,...\rangle$ be a sequence of words ($\mathcal{A}$ may be either finite or infinite.) We define $W_n(\mathcal{A})$ to be the set of all words $w \in X^*$ of length n which do not have any α_i as a factor.

Given two words w_1 and w_2, we say that w_2 *overlaps* w_1 *starting at k* if w_1 and w_2 can be written as $w_1 = uv$ and $w_2 = vw$, where v is a nonempty word and $l(u) = k - 1$. Given two sequences of words $\mathcal{A} = \langle \alpha_1, \alpha_2,...\rangle$, $\mathcal{B} = \langle \beta_1, \beta_2,...\rangle$, we say that $\mathcal{A}$ and $\mathcal{B}$ have the *consistent overlapping property* if

(1) For all i, $l(\alpha_i) = l(\beta_i)$.

(2) For all i, j, and k, α_i is a factor of α_j starting at k iff β_i is a factor of β_j starting at k.

(3) For all i, j, and k, α_i overlaps α_j starting at k iff β_i overlaps β_j starting at k.

We emphasize that the i and j in conditions (2) and (3) need not be distinct. For example, suppose $\alpha_1 = x_1^n$ = the word of length n with all letters equal to x_1. Then α_1 overlaps itself at $1, 2, \ldots, n$. Thus β_1 must overlap itself at $1, 2, \ldots, n$ and hence β_1 must equal x_i^n for some i. Thus the consistent overlapping property is a quite strong condition on $\mathscr{A}$ and $\mathscr{B}$. Nevertheless, there are nontrivial examples. For example if $X = \{x_1, \ldots, x_8\}$, then $\mathscr{A} = \{x_1x_1x_2x_2,\ x_3x_3x_4x_4,\ x_5x_5x_6x_6,\ x_7x_7x_8x_8\}$ and $\mathscr{B} = \{x_1x_2x_3x_4,\ x_2x_1x_4x_3,\ x_5x_6x_7x_8,\ x_6x_5x_8x_7\}$ have the consistent nonoverlapping property.

THEOREM 6. *Suppose $\mathscr{A} = \{\alpha_1, \alpha_2, \ldots\}$ and $\mathscr{B} = \{\beta_1, \beta_2, \ldots\}$ are sequences of distinct words from X^* with the consistent overlapping property, then for all n, $|W_n(\mathscr{A})| = |W_n(\mathscr{B})|$.*

Proof. We shall use the involution principle to define a bijection between $W_n(\mathscr{A})$ and $W_n(\mathscr{B})$ for fixed n. Given a word w, define $S_{\mathscr{A}}(w) = \{(\alpha_j, k) \mid \alpha_j$ is a factor of w starting at $k\}$. We order the elements of $S_{\mathscr{A}}(w)$ lexicographically, i.e., we define $(\alpha_{j_1}, k_1) < (\alpha_{j_2}, k_2)$ iff either (i) $j_1 < j_2$ or (ii) $j_1 = j_2$ and $k_1 < k_2$. We define and order $S_{\mathscr{B}}(w)$ similarly. Clearly, $W_n(\mathscr{A}) = \{w \in X^* \mid l(w) = n$ and $S_{\mathscr{A}}(w) = \phi\}$ and $W_n(\mathscr{B}) = \{w \in X^* \mid l(w) = n$ and $S_{\mathscr{B}}(w) = \phi\}$. This given, we can define the spaces A and B required for the application of Theorem 1. Let $A = \{(w, S) \mid w \in X^*, l(w) = n,$ and $S \subseteq S_{\mathscr{A}}(w)\}$ and $B = \{(w, S) \mid w \in X^*, l(w) = n,$ and $S \subseteq S_{\mathscr{B}}(w)\}$. The sign of a pair $\langle w, S\rangle$ in either A or B is $(-1)^S$. The involutions α and β are like the involutions in Theorem 2. That is, if $S_{\mathscr{A}}(w) \neq \phi$, then let $m_{\mathscr{A}}(w) = \max(S_{\mathscr{A}}(w))$ and define

$$\begin{aligned} \alpha(w, S) &= (w, S - \{m_{\mathscr{A}}(w)\}), &&\text{if} \quad m_{\mathscr{A}}(w) \in S_{\mathscr{A}}(w), \\ &= (w, S \cup \{m_{\mathscr{A}}(w)\}), &&\text{if} \quad m_{\mathscr{A}}(w) \notin S_{\mathscr{A}}(w). \end{aligned}$$

If $S_{\mathscr{A}}(w) = \phi$, then $\alpha(w, \phi) = (w, \phi)$, and β is defined similarly. Clearly, α and β are sign-reversing involutions with fixed point sets $W_n(\mathscr{A}) \times \phi$ and $W_n(\mathscr{B}) \times \phi$, respectively.

Finally, the sign preserving map $f\colon A \to B$ is defined as follows: Given a pair $(w, S) \in A$, $f(w, S) = (w', S')$, where w' is the word obained from w by replacing each factor α_j starting at k with $(\alpha_j, k) \in S$ by the factor β_j starting at k and $S' = \{(\beta_j, k) \mid (\alpha_j, k) \in S\}$. For example, suppose $X = \{x_1, x_2, x_3, x_4\}$, $\mathscr{A} = \{x_1x_2x_2, x_2x_3x_2\}$, and $\mathscr{B} = \{x_1x_2x_3, x_3x_4x_3\}$. Then $f(x_1x_2x_2x_3x_2x_1x_2x_3x_2,\ \{(x_1x_2x_2, 1),\ (x_2x_3x_2, 3),\ (x_2x_3x_2, 7)\}) = (x_1x_2x_3x_4x_3x_1x_3x_4x_3,\ \{(x_1x_2x_3, 1)(x_3x_4x_3, 3),\ (x_3x_4x_3, 7)\}$. Note that the consistent overlapping property of $\mathscr{A}$ and $\mathscr{B}$ ensure that w' is well defined. Then just as in Theorem 2, the iterated (α, β)-map $I\langle\alpha, \beta, f\rangle$ is a bijection between $W_n(\mathscr{A}) \times \{\phi\}$ and $W_n(\mathscr{B}) \times \{\phi\}$ which may be regarded as a bijection between $W_n(\mathscr{A})$ and $W_n(\mathscr{B})$. ∎

ACKNOWLEDGMENTS

We wish to gratefully acknowledge A. M. Garsia for many helpful conversations concerning partition theory in general and our work here in particular and to thank H. Wilf for his suggestion that our ideas should apply to the problems discussed in Section 3.

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